

# Exponential growth in two-dimensional topological fluid dynamics\*

Philip Boyland

October 10, 2012

## Abstract

This paper describes topological kinematics associated with the stirring by rods of a two-dimensional fluid. The main tool is the Thurston-Nielsen (TN) theory which implies that depending on the stirring protocol the essential topological length of material lines grows either exponentially or linearly. We give an application to the growth of the gradient of a passively advected scalar, the Helmholtz-Kelvin Theorem then yields applications to Euler flows. The main theorem shows that there are periodic stirring protocols for which generic initial vorticity yields a solution to Euler's equations which is not periodic and further, the  $L^\infty$  and  $L^1$ -norms of the gradient of its vorticity grow exponentially in time.

## 1 Introduction

Knots are an essential ingredient of three-dimensional topological fluid dynamics. Their presence as flow or field lines is a marker of a certain level of complexity in the system. A fundamental topological principle is that codimension-two is the knotting dimension, so circles can be knotted in 3D and two-spheres in 4D. In a two-dimensional flow, points are codimension two, but can they be knotted? Clearly not if we consider the static problem, but if we consider their motion, it is plausible that if they get sufficiently entangled during their evolution they can have implications for the surrounding flow field.

There is a substantial body of mathematics available to understand this situation, most prominently, the Thurston-Nielsen theory. This theory has many aspects and ramifications, but the part that is most directly applicable to two-dimensional fluid dynamics concerns the growth of material lines. Specifically, there are certain motions of points or rods in a fluid which imply that the length of a class of material lines must grow exponentially as they are advected by the fluid. Further, the theory gives many methods to detect and/or construct such motions as well as algorithms for computing the rate of exponential growth.

In a two-dimensional incompressible fluid when a material line is being stretched exponentially, then under the tangent map some vectors are growing and others are shrinking

---

\*For the proceedings of the IUTAM Symposium on Topological Fluid Mechanics II, Cambridge, UK, July 2012

exponentially. Heuristically, this has implications for a (typical) passively advected scalar: its gradients must also grow exponentially. For an Euler flow, the Helmholtz-Kelvin Theorem says that the vorticity is passively advected and thus its gradients are growing exponentially. However, to make these arguments precise, we need conditions on the advected scalar to make sure it has nontrivial gradients and more importantly, we must eliminate the possibility of “unfortunate coincidences” where the gradients often line up with a contracting direction in the fluid.

As is usual in Dynamical Systems Theory, these issues are dealt with by considering only generic scalars. A set is considered topologically large or generic if it is dense,  $G_\delta$ , *ie.* it is the intersection of a countable family of open, dense sets. Thus with the appropriate norm (and thus topology) on the set of scalar functions we only consider those in a carefully chosen dense,  $G_\delta$ -set. For the case of Euler flows, we use the fact that the initial vorticity determines the initial velocity field and so the initial vorticities “parameterize” the collection of Euler flows. Our results concern the behaviour of typical values of this “parameter”.

The second component of this paper studies the situation where a time-periodic stirring protocol gives rise to a time-periodic fluid motion. This component does not use the TN-theory but shares the two themes of growth rate of material lines and generic hypothesis. For a time-periodic fluid motion the presence of a time-periodic passively advected scalar implies that the flow is integrable and thus, in particular, the growth rate of advected material lines is at most linear. Once again, the Helmholtz-Kelvin Theorem yields an immediate application to Euler flows.

For more information on Thurston-Nielsen Theory see [Thu88] and [FLP91], for its dynamical systems applications see [Boy94], and for its fluid mechanical applications see [BAS00] and [TF06]. For many more references (and pictures of fluids being stirred by pseudo-Anosov protocols) see the papers by Thiffeault and Stremler in this volume. The theorems in this paper appear in [Boy05] in a somewhat different form; see that paper for details of the proofs.

## 2 Topological kinematics

### 2.1 Basic definitions and terminology

We first formalize the stirring of a planar body of fluid by moving rods. The *fluid region* is a smooth, one-parameter family of smooth, multi-connected, compact, planar domains denoted  $M_t$ . In this family the outer boundary is held fixed while the inner disks move. We shall always assume time-periodicity of the domains with period one, so  $M_{t+1} = M_t$  for all  $t$ . The moving inner regions are called the *stirrers*, and they are perhaps permuted each period.

Since we will be using a fair amount of dynamical systems terminology and what is called a flow in dynamical systems is called a steady flow in fluid mechanics, we shall adopt the terminology that a *fluid motion* is a smooth one-parameter family of diffeomorphisms,  $\phi_t : M_0 \rightarrow M_t$ , with  $\phi_0 = id$ . We may view  $\phi_t$  as Lagrangian fluid displacement map: the particle at  $\mathbf{x} \in M_0$  at time 0 is at  $\phi_t(\mathbf{x}) \in M_t$  at time  $t$ . However, at this point we are

making no assumption that the fluid or its *velocity field*

$$\mathbf{u}(\phi_t(\mathbf{x}), t) := \frac{\partial \phi_t}{\partial t}(\mathbf{x})$$

satisfies any particular equations. Because the fluid does not penetrate the moving boundary (expressed by  $\phi_t : M_0 \rightarrow M_t$ ) the velocity field satisfies the boundary conditions  $\mathbf{u} \cdot \mathbf{n}_i = \dot{B}_i \cdot \mathbf{n}_i$ , with  $B_i(t)$  the motion of the  $i^{th}$  boundary. The fluid motion is *incompressible or area-preserving* if  $\nabla \cdot \mathbf{u} = 0$ , or equivalently, the Jacobian of  $\phi_t$  is identically one,  $\det(D\phi_t) \equiv 1$ .

We have assumed that the stirring protocol is time-periodic. A special situation of importance is when the velocity field is also time-periodic with the same period. In this case we can study the time evolution using the *Poincaré or time-one map*,  $\phi_1 : M_0 \rightarrow M_0 = M_1$ , which satisfies  $\phi_n = \phi_1^n$ , with the superscript indicating repeated composition.

## 2.2 One-dimensional growth rates

A material line in the fluid is described by a smooth arc or simple closed curve (scc)  $\gamma$ . The main focus here is how the material line grows in length as it is passively transported by the fluid motion, which means we are analyzing the length of  $\phi_t \circ \gamma$  as a function of  $t$ . We will use two different ways to measure the length of the curve, one topological and the other metric.

### 2.2.1 Metric growth

Let  $\ell_t(\gamma)$  denote the length of the curve  $\gamma$  with respect to some smooth, periodic family of Riemannian metrics on the  $M_t$ . The *metric growth rate* of  $\gamma$  is the growth of

$$L_t^{met}(\gamma) := \frac{\ell_t(\phi_t \circ \gamma)}{\ell_0(\gamma)}$$

as a function of  $t$ . There are many methods available to quantify growth rate. The situation here is rather simple and we will usually just be bounding the growth above and/or below by simple functions, namely, by  $c\lambda^t$  with  $\lambda > 1$  for *exponential growth* and by  $ct$  for *linear growth*.

### 2.2.2 Topological growth

The *topological one-dimensional growth rate* is designed to depend on just the topology of the stirring process and to be independent of various details of the fluid motion. There are two main ideas. First, we restrict consideration to so-called essential arcs and scc which truly “see the topology” of the evolving regions. Second, at each moment in time we don’t compute the actual metric length of the evolving curve but rather we compute the shortest length amongst curves with the same topology. Imagine an advecting arc as elastic; at each time we will let it shrink back to the shortest length while maintaining the endpoints on the same boundary circle. Maintaining the curves topology means that in the shortening process it is not allowed to pass through the solid stirrers.

These ideas are more formally defined using homotopies. Two arcs  $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$  are *homotopic* in  $X$  if there exists a continuous map  $\Gamma : [0, 1] \times [0, 1] \rightarrow X$  with  $\Gamma(1, t) = \gamma_1(t)$  and  $\Gamma(0, t) = \gamma_2(t)$ . Thus  $\Gamma$  gives a continuous deformation from one arc to the other.

An *essential arc* is required to have its endpoints on the boundary and two essential arcs are homotopic if there is a homotopy  $\Gamma$  between them with the additional property that the homotopy keeps the endpoints on the boundary, so  $\Gamma(s, 0)$  and  $\Gamma(s, 1)$  are contained in the boundary for all  $s$ . Note that by continuity of  $\Gamma$  this means that  $\gamma_1$  and  $\gamma_2$  have their endpoints on the same pair of boundary circles. In addition, for essential arcs  $\gamma$  it is required that  $\gamma$  is not homotopic to an arc wholly contained in a single boundary circle. So, for example, an arc whose endpoints are on different boundary circles is always essential.

We let  $S^1$  denote the circle and so a scc in  $X$  is a continuous, injective map  $\gamma : S^1 \rightarrow X$ . Two scc  $\gamma_1, \gamma_2 : S^1 \rightarrow X$  are *homotopic* in  $X$  if there exists a continuous map  $\Gamma : [0, 1] \times S^1 \rightarrow X$  with  $\Gamma(1, t) = \gamma_1(t)$  and  $\Gamma(2, t) = \gamma_2(t)$ . An *essential scc* is one that is neither homotopic to a point nor homotopic to a boundary circle. Thus an essential scc encloses at least two but not all of the stirrers. All the arcs and scc shown in Fig. 1.(b) are essential.

In both cases we denote the set of curves homotopic to  $\gamma$ , *ie.* its homotopy class, as  $[\gamma]$  and the topological length  $L^{top}(\gamma)$  of a curve will be the least length among curves in its homotopy class

$$L^{top}(\gamma) := \min\{\ell(\sigma) : \sigma \in [\gamma]\}.$$

The *topological growth rate of the class*  $[\gamma]$  is the growth of

$$L_t^{top}(\gamma) := \frac{L^{top}(\phi_t \circ \gamma)}{L^{top}(\gamma)}.$$

Thus to compute the topological growth rate we evolve the curve forward for time  $t$  and then measure the least length in its homotopy class. An crucial property for essential curves  $\gamma$  is that

$$L_t^{met}(\gamma) \geq L_t^{top}(\gamma). \quad (2.1)$$

### 2.3 Topological entropy

A fundamental result for two-dimensional iterated  $C^\infty$ -diffeomorphisms is that the maximum exponential metric growth rate of arcs is equal to the topological entropy ([NP93], [New91]). Note the potential confusion with the current terminology, namely, the maximal *metric* one-dimensional growth is equal to the *topological* entropy. This entropy is so-called since it may be defined using growth rates of distances using a topological metric, in contrast to the measure-theoretic (or metric) entropy which requires an invariant measure. The variational principle says that the topological entropy is the supremum of the measure-theoretic entropies over all invariant measures: however, for curves, the topological growth is a lower bound for the metric growth. With all this potential confusion we hopefully clarify matters by emphasizing that our main concern here is the exponential growth of the lengths of one-dimensional curves.

In analogy with the result for iterated diffeomorphisms, for a general two-dimensional fluid motion  $\phi_t$  define its *topological entropy* as

$$h_{top}(\phi_t) = \sup \left\{ \limsup_{t \rightarrow \infty} \frac{\log(L_t^{met}(\gamma))}{t} : \gamma \text{ is a smooth arc} \right\}.$$

In accord with equation (2.1), if some essential curve  $\gamma$  has  $L_t^{top}(\gamma) \geq C\lambda^t$  for some constants  $C > 0$  and  $\lambda > 1$ , then  $h_{top}(\phi_t) \geq \log(\lambda) > 0$ .

## 2.4 Isotopy and braids

As noted in Subsection 2.2.2, the topological growth rate of curves depends only on the rough topology of the stirrer motion. This motion, in turn, determines what is called the isotopy class of the map  $\phi_1$ . More precisely, two homeomorphisms  $f, g : M_0 \rightarrow M_0$  are *isotopic* if there is a continuous family of homeomorphisms  $h_t$  deforming one to the other with  $h_0 = f$  and  $h_1 = g$ . Given an essential arc  $\gamma$ , applying the isotopy between  $f$  and  $g$  yields a homotopy  $H_t = h_t \circ \gamma$  between  $f \circ \gamma$  and  $g \circ \gamma$ . Thus we see that isotopic homeomorphisms give the same topological growth rate of an essential curve  $\gamma$ .

The isotopy class of a stirring protocol can be visualized and in fact characterized by the space-time trace of the stirrers or their *braid*. The collection of braids forms a group and the algebra of the braid gives a convenient method to compute properties associated with the trichotomy of the next subsection.

Another comment that will be relevant later is that because our stirring protocols are periodic,  $\phi_1$  is isotopic to  $\phi_n$  for all  $n \in \mathbb{N}$ , even when the velocity field is not periodic.

## 2.5 The Thurston-Nielsen trichotomy

For the purposes of this paper, the *Thurston-Nielsen theory* classifies stirred fluid motions and their isotopy classes based upon the rate of their topological one-dimensional growth: is it exponential or linear.

**Theorem 2.1 (Thurston-Nielsen Trichotomy)** *Let  $M_t$  be periodic stirring protocol with fluid motion  $\phi_t$ . Then either*

- (1) *PseudoAnosov (pA): there exist constants  $\lambda > 1$  (the dilation) and  $0 < C_1 < C_2$  such that for every essential curve  $\gamma$ ,*

$$C_1 \lambda^t \leq L_t^{\text{top}}(\gamma) \leq C_2 \lambda^t,$$

*and thus  $h_{\text{top}}(\phi_t) \geq \log(\lambda) > 0$ .*

- (2) *Finite order (fo): there exists a constant  $K > 0$  such that for every essential curve  $\gamma$ ,*

$$L_t^{\text{top}}(\gamma) < K t.$$

- (3) *Reducible case: (roughly stated)  $M_0$  splits into  $\phi_1$ -invariant subsurfaces on which 1. or 2. holds.*

We shall call a stirring protocol finite order, pseudoAnosov, or reducible according to its TN-type. A critical feature of the TN-trichotomy is that in the finite order and pA cases, every essential curve has the same basic topological growth rate. The pseudoAnosov case is the most useful for applications because this rate is exponential.

It is important to emphasize that the TN theory only concerns topological growth rates and thus it only gives bounds for the more physical metric growth. Many or perhaps most of the fluid motions arising from the same stirring protocol (*ie.* in the same isotopy class) could have a greater growth rates. In particular, a finite order stirring protocol of a real

fluid will almost certainly have regions of exponential metric growth while its topological one-dimensional growth is always linear.

It is also important to note that the theorem was first proved in the much broader context of isotopy classes of surface homeomorphisms. It is usually framed in terms of the existence of a special map, the Thurston-Nielsen representative, which is present in each isotopy class. When this special representative map is pseudoAnosov, it has a wealth of nice properties including a Markov partition which yields a symbolic model of a mixing subshift of finite type. This has many implications, for example, the pseudoAnosov map is ergodic and mixing with respect to Lebesgue measure and has a dense orbit and its set of periodic orbits is also dense. Handel’s Isotopy Stability Theorem shows that most of the dynamical properties of the pseudoAnosov map are present in any other map in the isotopy class, though they could exist in a “small” invariant set ([Han85]).

### 3 Passive advection of scalars

In this section we give two results concerning the behaviour of advected scalars under a stirring protocol. The results continue to be strictly kinematic. Given a fluid motion  $\phi_t$ , a function  $\alpha : M_t \times \mathbb{R} \rightarrow \mathbb{R}$  is called a *passively advected scalar* if it is constant on trajectories,  $\alpha_t(\phi_t(\mathbf{x})) = \alpha_0(\mathbf{x})$ , or equivalently,  $\partial \alpha_t(\phi_t(x))/\partial t = 0$ , where we have written  $\alpha_t(\mathbf{x})$  for  $\alpha(\mathbf{x}, t)$ . In the language of global analysis one says that  $\alpha_t$  is the *push forward* of  $\alpha_0$  and writes  $(\phi_t)_*(\alpha_0) = \alpha_t$ , with  $(\phi_t)_*(\alpha_0) = \alpha_0 \circ (\phi_t)^{-1}$ .

Now for any function  $f : M_0 \rightarrow \mathbb{R}$  we may obtain a passively advected scalar simply by defining  $\alpha_t := (\phi_t)_*(f)$ , and so in this sense only the initial configuration  $\alpha_0$  matters. For example, if  $\alpha_0$  represents the initial concentration of dye in a fluid,  $\alpha_t = (\phi_t)_*(\alpha_0)$  is the concentration after time  $t$ . On the other hand, in a physical fluid there may be scalar quantities of importance which at each time  $t$  are computed from the velocity field. In this case the advected scalar represents a conserved quantity having physical meaning. For example, in two dimensions the curl,  $\omega_t = \nabla \times \mathbf{u}$ , is a scalar which is passively advected in an Euler flow.

#### 3.1 Consequences of a pseudoAnosov protocol

For a fluid motion determined by a pseudoAnosov protocol we know from the TN-trichotomy and equation (2.1) that the metric length of essential curves is growing exponentially fast. This means that tangent vectors to these curves must be growing in length exponentially under the action of the space derivative of the fluid motion  $D\phi_t$ . This implies that an eigenvalue of  $D\phi_t$  is growing exponentially. For a passively advected scalar  $\alpha_t$ , since  $\alpha_t = \alpha_0 \circ (\phi_t)^{-1}$ , we have  $\nabla \alpha_t = \nabla \alpha_0 (D\phi_t)^{-1}$ . If the fluid motion is incompressible then  $\det(D\phi_t) = 1$  and so  $(D\phi_t)^{-1}$  also has an eigenvalue growing exponentially. Thus as long as there is not a unfortunate coincidence where  $\nabla \alpha_0$  stays aligned with the stable eigen-direction of  $(D\phi_t)^{-1}$ , we have that  $|\nabla \alpha_t|$  is growing exponentially.

There are a number of issues involved with making this argument rigorous; the locations in  $\gamma$  where there is tangential growth will be shifting in time and the unfortunate coincidence could actually happen, especially if one is working in a restricted class of physical interest. One way around these difficulties is to take a more geometric global viewpoint.

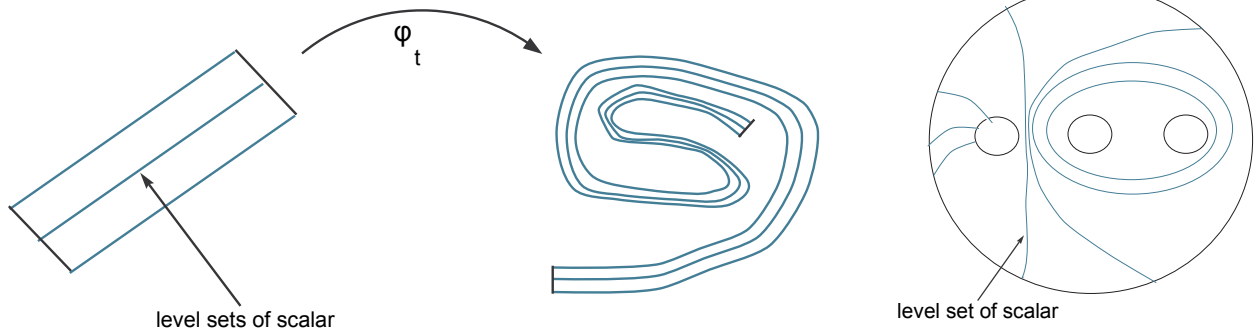


Figure 1: (a) Proof of Theorem 3.1

(b) Proof of Theorem 3.2

**Theorem 3.1** *Let  $M_t$  be a time-periodic stirring protocol of pA type with incompressible fluid motion  $\phi_t$ . If  $\alpha_t$  is a passively advected scalar such that its initial state  $\alpha_0$  is a generic  $C^2$ -function, then there are positive constants  $c, c'$  so that*

$$\sup_{\mathbf{x} \in M_0} |\nabla \alpha_t(\mathbf{x})| \geq c\lambda^t \quad \text{and} \quad \int_{M_t} |\nabla \alpha_t(\mathbf{x})| \geq c'\lambda^t$$

for all  $t \in \mathbb{R}$ , where  $\lambda > 1$  is the dilation of the pseudoAnosov protocol.

Here are the main ideas in the proof. First, find a  $C^2$ -open, dense set  $\mathcal{G}$  inside the Morse functions (functions with nondegenerate critical points) on  $M_0$  so that  $\alpha_0 \in \mathcal{G}$  implies that  $\alpha_0$  has a band of regular inverse images which are essential arcs or circles. The pA protocol forces a stretch in length by  $\lambda^t$ . This coupled with area preservation and transport of vorticity force the level sets of  $\alpha_t = (\phi_t)_*(\alpha_0)$  to bunch up in the transverse direction, which causes  $\|\nabla \alpha_t\|_\infty \rightarrow \infty$  like  $\lambda^t$ . See Fig. 1.(a)

### 3.2 Time-periodic fluid motions

Now consider the special situation when the velocity field of the fluid motion  $\phi_t$  is time-periodic with period one and so  $\phi_1$  is a Poincaré map. In addition, we assume the existence of a passively advected scalar that is also time-periodic. This again is rather special, but if the advected scalar depends on the velocity field (eg its curl), it is automatically time-periodic when the velocity field is. In this situation we show that if the scalar field is typical in the appropriate sense, then there is at most linear growth of the metric length of material lines.

The main observation required is that given a time-periodic passively transported scalar  $\alpha_t = \alpha_{t+1}$  and the Poincaré map  $\phi_1$ , the initial configuration  $\alpha_0$  satisfies for every  $\mathbf{x}$

$$\alpha_0(\mathbf{x}) = \alpha_1(\phi_1(\mathbf{x})) = \alpha_0(\phi_1(\mathbf{x})).$$

Thus  $\alpha_0$  is what is called an *integral of motion* for the map  $\phi_1$  and so as long as  $\alpha_0$  is sufficiently nondegenerate, its existence precludes chaotic dynamics.

**Theorem 3.2** *Let  $M_t$  be a time-periodic stirring protocol with time-periodic fluid motion  $\phi_t$  and time-periodic transported scalar  $\alpha_t$ . If the initial state of the scalar  $\alpha_0$  has finitely many critical points (for example, is  $C^\omega$  and nonconstant or is  $C^2$ -generic), there exists a constant  $K > 0$  so that*

$$L_t^{\text{met}}(\gamma) \leq Kt$$

*for all arcs and scc  $\gamma$ . Thus the metric one-dimensional growth is linear, and the Poincaré map  $\phi_1$  has zero topological entropy and zero Lyapunov exponents almost everywhere.*

The geometric heart of the proof is this: the level sets of  $\alpha_0$  must be preserved by  $\phi_1$  and for a generic function  $\alpha_0$  the contours  $\alpha_0(\mathbf{x}) = c$  are smooth arcs and circle for typical  $c \in \mathbb{R}$ . Thus the dynamics of  $\phi_1$  consists of one-dimensional invariant subsets. Injective maps of arcs and circles yield very simple dynamics with no dynamical entropy and using this in the entire fluid domain we have only linear metric growth of arcs.

### 3.3 Combining the two results

Putting these two kinematic results together yields an observation of relevance to Euler flows. If a time-periodic fluid motion has a generic advected scalar field that depends on the velocity field and thus is also time-periodic we know from Theorem 3.2 that there is linear metric one-dim growth of all curves (material lines). On the other hand, we know from Theorem 3.1 that a pseudoAnosov stirring protocol always causes exponential stretching of some material lines. These clearly can't coexist and so if a fluid motion is stirred by a pA protocol and has a passively advected generic scalar then either the scalar is not dependent directly on the velocity field or if it is, then the fluid motion is not time-periodic.

## 4 Euler fluid motions

### 4.1 Basic definitions and results

Now we assume that the velocity field  $\mathbf{u}(\mathbf{x}, t)$  of the fluid motion  $\phi_t$  satisfies the incompressible, constant density ( $\rho \equiv 1$ ), Euler equation

$$\frac{D\mathbf{u}}{Dt} = -\nabla p_t, \quad \text{div}(\mathbf{u}) = 0,$$

with slip boundary conditions on the moving boundary. In this case  $\phi_t$  is called an *Euler fluid motion*. Kozonoi has shown that for the class of two-dimensional problems with moving boundary considered here there are classical solutions ([Koz85]). The results in this paper are all predicated on the assumption that there is a global strong solution with the regularity of the initial data: our goal is to analyze its dynamics.

Recall that for two-dimensional, divergence-free velocity fields a classical result says that the curl coupled with the circulations around the boundary components and the boundary conditions  $\mathbf{u} \cdot \mathbf{n}_i = \dot{B}_i \cdot \mathbf{n}_i$  determine the field completely. In a certain sense we will view the collection of Euler fluid motions as being “parameterized” by possible initial vorticities and all our results concern dynamics associated with typical values of these “parameters”.

The Helmholtz-Kelvin Theorem (1890's) is perhaps the most important result about two-dimensional Euler fluid motions: an incompressible fluid motion is Euler if and only if its



vorticity is passively transported and circulations around all smooth simple closed curves  $C$  are preserved, or

$$\frac{d}{dt} \oint_{\phi_t(C_i)} \mathbf{u} \cdot d\mathbf{r} = 0$$

for each boundary circle  $C_i$ . The preservation of boundary circulation is a necessary feature in multi-connected domains.

## 4.2 An Exponential Growth Theorem

Now we consider an Euler fluid motion under the influence of a pseudoAnosov stirring protocol. Theorem 3.1 used with the Helmholtz-Kelvin Theorem immediately yields:

**Theorem 4.1** *Let  $M_t$  be a time-periodic stirring protocol of pA type with Euler fluid motion  $\phi_t$ . If the initial vorticity  $\omega_0$  is a generic  $C^2$ -function, then there are positive constants  $c, c'$  so that*

$$\sup_{\mathbf{x} \in M_0} |\nabla \omega_t(\mathbf{x})| \geq c\lambda^t \quad \text{and} \quad \int_{M_t} |\nabla \omega_t(\mathbf{x})| \geq c'\lambda^t \quad (4.1)$$

for all  $t \in \mathbb{R}$ , where  $\lambda > 1$  is the dilation of the pA protocol. Thus  $\|\Delta \mathbf{u}(\mathbf{x}, t)\|_\infty = \|\nabla \omega_t\|_\infty \rightarrow \infty$  and  $\|\mathbf{u}_t\|_{C^2} \rightarrow \infty$ , all like  $\lambda^t$ .

There are numerous results in the stability literature concerning growth of  $|\nabla \omega_t|$  for perturbations of two-dimensional steady Euler fluid motions. Yudovich ([Yud74], [Yud00]) and others have shown linear growth and Arnol'd ([Arn72]), Friedlander and Vishik ([FV92]) and others have shown the importance of exponential growth. The same basic mechanism is in play here as in those results, namely, the growth of  $\nabla \omega$  under the influence of the eigenvectors of the tangent map  $D\phi_t$  as discussed at the beginning of Subsection 3.1.

## 4.3 Speculations on applications to more general Euler flows

We make a few very speculative remarks on how the exponential growth induced by a pseudoAnosov stirring protocols might have implications for general Euler flow. In particular, it provides a topological perspective on a version of the *Yudovich Hypothesis/Conjecture*: for generic initial vorticity a two-dimensional Euler fluid motion satisfies the exponential growth in equation (4.1) (cf. [MSY08]).

There are two main ingredients in this topological perspective. The first is that the exponential growth of material lines caused by the stirrers could just as well be achieved by advecting points in the fluid ([Bow78], [GTF05]). The TN-trichotomy as stated above requires these “virtual stirrers” or “ghost rods” to be periodic points. However, it is clear that an aperiodic “tangle” of points evolving can also force exponential topological growth of a class of material lines. The second ingredient would say that for typical initial vorticity a two-dimensional Euler fluid motion always has such a collection of fluid trajectories. If both these ingredients were known, virtually the same proof as Theorem 4.1 would yield the version of the Yudovich Conjecture just given. While both ingredients seem quite reasonable, rigorous results seem a long way off. The topological ingredient is probably tractable but determining even a part of the dynamical evolution of a typical two-dimensional Euler fluid motion requires tools still to be developed.

#### 4.4 A linear metric growth theorem

Because the vorticity  $\omega_t$  is passively transported scalar for an Euler fluid motion as in Subsection 3.2 it follows that when the fluid motion is time-periodic, the initial configuration  $\omega_0$  is an integral of motion for the Poincaré map  $\phi_1$ . Brown and Samelson ([BS94]) observed using a theorem of Moser that if the integral of motion  $\omega_0$  is real analytic, then  $\phi_1$  can't have a transverse homoclinic intersection. More generally using Theorem 3.2 and the Helmholtz-Kelvin Theorem we have:

**Theorem 4.2** *Let  $M_t$  be a time-periodic stirring protocol with time-periodic Euler fluid motion  $\phi_t$ . If the initial vorticity  $\omega_0$  has finitely many critical points (for example, is  $C^\omega$  and nonconstant or is  $C^2$ -generic), there exists a constant  $K > 0$  so that*

$$L_t^{met}(\gamma) \leq Kt$$

*for all arcs and scc  $\gamma$ . Thus the metric one-dimensional growth is linear, and the Poincaré map  $\phi_1$  has zero topological entropy and zero Lyapunov exponents almost everywhere.*

The crucial features of the theorem are the time-periodicity of the Euler fluid motion and its two-dimensionality. Thus a similar result holds in any two-dimensional fluid region. It is also important to note what the theorem does not say. It does not say that the generic initial vorticity gives rise to a time-periodic Euler flow, and it does not say that among initial vorticities which do give rise to time-periodic Euler fluid motions linear growth is typical. Understanding these issues requires a knowledge of the size of the set of time-periodic Euler fluid motions which are not steady. This is ill-understood at present. There are reasons to believe that, in fact, generic initial vorticity never gives rise to a time-periodic Euler fluid motion and Theorem 4.2 could be useful in proving this.

#### 4.5 A dichotomy for time-periodic 2D Euler flows

Theorem 4.2 is similar to and inspired by Arnol'd's result ([Arn65]) on steady, three dimensional  $C^\omega$ -Euler flows which we frame as a dichotomy:

- (1) For *typical* vorticity the preservation of the Bernoulli function  $p + \|\mathbf{u}\|^2/2$  implies that the flow is integrable and typical orbits are confined to invariant tori and annuli and the flow thus has zero entropy and zero Lyapunov exponents almost everywhere.
- (2) For the *atypical* case of a Betrami flow where the curl is parallel to the velocity field, one may have measures with positive entropy, positive Lyapunov exponents, etc.

For *time-periodic, two-dimensional* Euler fluid we know from Theorem 4.2 that in analogy with 1., for typical initial vorticity there is integrability and linear metric one-dimensional growth. For an analog of the atypical case 2., perhaps the simplest situation is when the vorticity is constant everywhere. In this case one always has time-periodic solutions for every time-periodic stirring protocols using classical potential theory: if  $C$  is a given constant and  $(\Gamma_0, \Gamma_1, \dots, \Gamma_m) \in \mathbb{R}^{m+1}$  with  $\sum \Gamma_i = 0$  is a vector of circulations, then there exists a unique time-periodic Euler fluid motion  $(M_t, \mathbf{u}_t)$  with  $\omega_t \equiv C$  for all time  $t$  and  $\oint_{B_i(t)} \mathbf{u} \cdot d\mathbf{r} = \Gamma_i$ , for  $i = 0, \dots, m$ . The proof is essentially to solve the Poisson equation  $\Delta \Psi = C$  for the needed

stream function  $\Psi$  at each time  $t$  and these stream functions depend on time-periodic data and so are time-periodic as are the velocity fields  $\nabla^\perp \Psi$ .

Now if we assume the stirring protocol is pseudoAnosov with dilation  $\lambda$ , the resulting fluid motion satisfies  $L^{met}(\gamma) \geq L^{top}(\gamma) \geq c\lambda^t$  for any essential curve  $\gamma$ . This implies that the Poincaré map  $\phi_1$  has  $h_{top}(\phi_1) \geq \log(\lambda) > 0$ , ergodic invariant measures with positive metric entropy and thus positive Lyapunov exponents and more . . . . Thus one dichotomy for time-periodic solutions to Euler's equation in a time-periodic moving domain is:

- (1) For *typical* initial vorticity there is integrability and the linear growth theorem above.
- (2) For the *atypical* case of constant vorticity classical potential theory yields time-periodic Euler fluid motions and with pseudoAnosov protocols these have chaotic dynamics.

However, once again we emphasize that the dichotomy is strictly conditional. It says nothing about the existence or prevalence of non-steady, time-periodic Euler fluid motions.

#### 4.6 An energy bound

Conservation of energy is a fundamental property of Euler fluid motions in stationary domains. For stirring protocols  $M_t$  a simple computation yields that the total energy  $E = \frac{1}{2} \|\mathbf{u}\|_2^2$  is evolving so that

$$\frac{dE}{dt} = - \sum \oint_{\phi_t(C_i)} p \dot{B}_i \cdot d\mathbf{n}_i.$$

This reflects the fact that the fluid can do work on the stirrers and vice versa and so the long term behaviour of the energy in a stirred Euler fluid is initially unclear especially in light of the possible exponential growth of  $\|\nabla \omega_t\|_1$ . We sketch an argument whose main ideas are due to Steve Childress which shows that for time-periodic stirring protocols the energy is uniformly bounded as  $t \rightarrow \infty$ .

Assume now that the Euler fluid motion  $\mathbf{u}$  has smooth initial vorticity  $\omega_0$  and circulation around the  $i^{th}$  boundary equal to  $\Gamma_i$ . By Helmholtz-Kelvin for all times  $t$ ,  $\mathbf{u}(\mathbf{x}, t)$  has the same circulations and further,  $|\omega_t| \leq \Omega := \max(|\omega_0|)$ . We now assume for simplicity that the outer boundary of  $M_t$  is the unit circle  $S^1$  and so  $M_t$  is contained in the unit disk  $D$ . We further assume that each stirrer is circular with area  $a$ . Now fix a time  $t$  and suppress dependence on it. We decompose  $\mathbf{u}$  into pieces whose energy will be bounded separately.

First, extend  $\omega$  to a function  $\hat{\omega}$  on  $\mathbb{R}^2$  with  $\hat{\omega} \equiv 0$  outside  $D$  and  $\hat{\omega} \equiv \Gamma_i/a$  inside the  $i^{th}$  stirrer. Using Biot-Savart we define

$$\mathbf{v}(\mathbf{x}) = \frac{1}{2\pi} \iint_D \frac{\hat{\omega}(\mathbf{y})(\mathbf{y} - \mathbf{x})^\perp}{|\mathbf{y} - \mathbf{x}|^2} d\mathbf{y}. \quad (4.2)$$

Thus  $\mathbf{v}$  has vorticity  $= \omega$  in  $M$ , has circulations  $\Gamma_i$ , and  $|\mathbf{v}| \leq \hat{\Omega}$  in  $M$  where  $\hat{\Omega} = \max\{\Omega, |\Gamma_i|\}$  using

$$\max_{\mathbf{x} \in M} \left\{ \frac{1}{2\pi} \iint_D \frac{1}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} \right\} = 1. \quad (4.3)$$

This yields  $\|\mathbf{v}\|^2 \leq \pi \hat{\Omega}^2$ .

Next, let  $h$  be the harmonic function on  $M$  with  $\partial h / \partial \mathbf{n} = -\mathbf{v} \cdot \mathbf{n}_i$  on each boundary circle  $C_i$  and  $\nabla h$  has zero circulation around all the boundaries. Finally, let  $g$  be the harmonic function on  $M$  with  $\partial g / \partial \mathbf{n} = \dot{B}_i$  on each boundary circle  $C_i$  and  $\nabla g$  has zero circulation around all the boundaries.

We now have that  $\mathbf{v} + \nabla h + \nabla g$  has vorticity  $\omega$ , circulations equal to  $\Gamma_i$ , and the normal component of its velocity on the boundary circle  $C_i$  is  $\dot{B}_i \cdot \mathbf{n}_i$  and so it is equal to  $\mathbf{u}$ . Because  $(\mathbf{v} + \nabla h) \cdot \mathbf{n}_i = 0$  on every boundary circle  $C_i$  and is divergence free we have that  $(\mathbf{v} + \nabla h) \perp \nabla h$  and  $(\mathbf{v} + \nabla h) \perp \nabla g$  working in  $L^2$ . Thus twice the energy of  $\mathbf{u}$  is

$$\|\mathbf{u}\|^2 = \|\mathbf{v} + \nabla h + \nabla g\|^2 = \|\mathbf{v} + \nabla h\|^2 + \|\nabla g\|^2 = \|\mathbf{v}\|^2 - \|\nabla h\|^2 + \|\nabla g\|^2 \leq \|\mathbf{v}\|^2 + \|\nabla g\|^2$$

Now we reintroduce the time dependence and do the decomposition of  $\mathbf{u}(\mathbf{x}, t)$  into three parts at each time  $t$ . By Helmholtz-Kelvin the circulations for  $\mathbf{u}(\mathbf{x}, t)$  and thus  $\mathbf{v}(\mathbf{x}, t)$  are the same for all  $t$  as is the value of  $\hat{\Omega}$ . Further, the bound using equation (4.3) is independent of  $t$  and so for all  $t$ ,  $\|\mathbf{v}(\mathbf{x}, t)\|^2 \leq \pi \hat{\Omega}^2$ . Finally, the data determining  $g(\mathbf{x}, t)$  is the position and velocity of the boundaries and so  $g$  is periodic in  $t$ , and we get  $\|\nabla g(\mathbf{x}, t)\|^2$  bounded above for all time by its supremum over one cycle, completing the time-independent bound on the energy of  $\mathbf{u}(\mathbf{x}, t)$ .

## References

- [Arn65] Vladimir Arnol'd. Sur la topologie des écoulements stationnaires des fluides parfaits. *C. R. Acad. Sci. Paris*, 261:17–20, 1965.
- [Arn72] V. I. Arnol'd. Notes on the three-dimensional flow pattern of a perfect fluid in the presence of a small perturbation of the initial velocity field. *J. Appl. Math. Mech.*, 36:236–242, 1972.
- [BAS00] P. L. Boyland, Hassan Aref, and Mark A. Stremler. Topological fluid mechanics of stirring. *J. Fluid Mech.*, 403:277–304, 2000.
- [Bow78] Rufus Bowen. Entropy and the fundamental group. In *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*, volume 668 of *Lecture Notes in Math.*, pages 21–29. Springer, Berlin, 1978.
- [Boy94] P. Boyland. Topological methods in surface dynamics. *Topology Appl.*, 58(3):223–298, 1994.
- [Boy05] P. Boyland. Dynamics of two-dimensional time-periodic Euler fluid flows. *Topology Appl.*, 152(1):87–106, 2005.
- [BS94] M. Brown and R. Samelson. Particle motion in vorticity- conserving, two-dimensional incompressible flow. *Phys. Fluids*, 6:2875–2876, 1994.
- [FLP91] A. Fathi, F. Laudenbach, and V. Poenaru. *Travaux de Thurston sur les surfaces*. Société Mathématique de France, Paris, 1991. Séminaire Orsay, Reprint of *Travaux de Thurston sur les surfaces*, Soc. Math. France, Paris, 1979, Astérisque No. 66-67 (1991).

- [FV92] Susan Friedlander and Misha M. Vishik. Instability criteria for steady flows of a perfect fluid. *Chaos*, 2(3):455–460, 1992.
- [GTF05] E. Gouillart, J.-L. Thiffeault, and M. D. Finn. Topological mixing with ghost rods. *Physical Review E*, 73(3):03631–036318, 2005.
- [Han85] Michael Handel. Global shadowing of pseudo-Anosov homeomorphisms. *Ergodic Theory Dynam. Systems*, 5(3):373–377, 1985.
- [Koz85] Hideo Kozono. On existence and uniqueness of a global classical solution of the two-dimensional Euler equation in a time-dependent domain. *J. Differential Equations*, 57(2):275–302, 1985.
- [MSY08] Andrey Morgulis, Alexander Shnirelman, and Victor Yudovich. Loss of smoothness and inherent instability of 2D inviscid fluid flows. *Comm. Partial Differential Equations*, 33(4-6):943–968, 2008.
- [New91] Sheldon E. Newhouse. Entropy in smooth dynamical systems. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 1285–1294, Tokyo, 1991. Math. Soc. Japan.
- [NP93] Sheldon Newhouse and Thea Pignataro. On the estimation of topological entropy. *J. Statist. Phys.*, 72(5-6):1331–1351, 1993.
- [TF06] Jean-Luc Thiffeault and Matthew D. Finn. Topology, braids and mixing in fluids. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 364(1849):3251–3266, 2006.
- [Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.
- [Yud74] V. I. Yudovich. On the loss of smoothness of the solutions of the euler equations. *Dynamics of Continuous Media (Dynamika Spolshnoy Sredi, Novosibirsk)*, 66:71–78, 1974.
- [Yud00] V. I. Yudovich. On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid. *Chaos*, 10(3):705–719, 2000.

Philip Boyland  
 Dept. of Mathematics  
 University of Florida  
 Little Hall  
 Gainesville, FL 32605-8105  
 boyland@math.ufl.edu